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Original article

SOME REMARKS ON BLASCHKE TYPE PRODUCTS IN AREA NEVANLINNA TYPE SPACES IN THE UNIT DISK

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Abstract. The intention of this paper is to introduce and study certain new analytic spaces in disk and to show that certain Blaschke type products belong to such so called new large Nevanlinna type classes in the unit disk. These results extend and complement some previously known assertions of this type obtained earlier by other authors. Our result may be used to get parametric representations of these large spaces of area Nevanlinna type via infinite B_α products.

Keywords: Blaschke type infinite products, area Nevanlinna type spaces, Nevanlinna characteristic, parametric representations, analytic function

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1. INTRODUCTION, BASIC DEFINITIONS AND HISTORY OF PROBLEMS

Assuming that $\mathbf{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disk of the finite complex plane $\mathbb{C}$, $\mathbf{T}$ is the boundary of $\mathbf{D}$, $\mathbf{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $H(\mathbf{D})$ is the space of all functions holomorphic in $\mathbf{D}$, we introduce the following classes of functions

$$N_\alpha^\infty(\mathbf{D}) = \left\{f \in H(\mathbf{D}): T(\tau, f) \le C_f(1-\tau)^{-\alpha},\ 0 \le \tau < 1,\ \alpha \ge 0\right\},$$

where $T(\tau, f)$ is classical Nevanlinna characteristic defined by $T(\tau, f) = \frac{1}{2\pi}\int_{\mathbf{T}} \log^+|f(\tau\xi)|d\xi$, where $a^+ = \max\{0, a\}, a \in \mathbb{R}$, (see [1]). It is obvious that if $\alpha = 0$ then $N_\alpha^\infty(\mathbf{D}) = N(\mathbf{D})$, where $N(\mathbf{D})$ is the well known classical Nevanlinna class (see [1], [2], [3]).

Let $f \in H(\mathbf{D})$, then we define

$$M_p(f, r) = \frac{1}{2\pi}\left(\int_T |f(r\xi)|^p dm(\xi)\right)^{\frac{1}{p}},\ r \in (0,1),\ p \in (0, \infty),$$

where by $m(\xi)$ we denote the normalized Lebesgue measure on $\mathbf{T}$. Also, by $m_2(\xi)$ we denote standard normalized Lebesques area measure.

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Everywhere below, by $n_f(t) = n(t)$ we denote the quantity of zeros of an analytic function f in the unit disk $|z| \le t < 1$ and by $Z(X)$ the zero set of an analytic class $X, X \subset H(\mathbf{D})$. By let $\{z_k\}_{k=1}^{\infty}$ be a sequence of numbers from $\mathbf{D}$, below we mean that $\{z_k\}_{k=1}^{\infty}$ is an arbitrary sequence from unit disk enumerated by it is growth $(|z_k| \le |z_{k+1}| \le \dots)$ according to it is multiplicity.

Let further

$$N_{\alpha,\beta}^{p}(\mathbf{D}) = \left\{ f \in H(\mathbf{D}) \colon \int_0^1 \left(\int_{|z|\le R} (\ln^+|f(z)|)(1-|z|)^{\alpha} dm_2(z) \right)^p (1-R)^{\beta} dR < \infty \right\};$$

$$N_{\alpha,\beta_1}^{\infty,p}(\mathbf{D}) = \left\{ f \in H(\mathbf{D}) \colon \sup_{0\le R<1} \left(\int_0^R \left(\int_T \ln^+|f(z)|\, d\xi \right)^p (1-|z|)^{\alpha} d|z| \right) (1-R)^{\beta_1} < \infty \right\},$$

where it is assumed that $\alpha > -1,\ \beta > -1,\ \beta_1 \ge 0$ and $0 < p < \infty$, and let

$$N_{p,\gamma,\beta}(\mathbf{D}) = \left\{ f \in H(\mathbf{D}) \colon \int_0^1 (1-|z|)^{\beta} \left(\sup_{0<\tau<|z|} T(\tau,f)(1-\tau)^{\gamma} \right)^p d|z| < \infty \right\},$$

where $\gamma \ge 0,\ \beta > -1,\ p \in (0,\infty)$.

We refer for basic properties of these new large are Nevanlinna spaces to [4] – [8]. We note that in these papers various results on zero sets and parametric representations can also be seen. Note similar, but less general results can be seen in various papers of various authors, we refer, for example, to [2], [3], [9], [10], [12].

Note that various properties of $N_{\alpha,0}^{\infty,p}(\mathbf{D}) = S_{\alpha}^{p}$ are studied in [9]. In particular, the works [2], [9] give complete descriptions of zero sets and parametric representations of $N_{\alpha,0}^{\infty,p}(\mathbf{D})$ (in [2] for $p = 1$). Thus it is natural to consider the problem of extension of these important results to all $N_{\alpha,\beta_1}^{\infty,p}(\mathbf{D})$ analytic classes.

It is not difficult to verify that all the above mentioned area Nevanlinna analytic classes are topological vector spaces with complete invariant metric. We note that the mentioned problem of parametric representation has various applications and are important in function theory (see [2], [3], [11]).

Solution of many problems for example the existence of radial limits is based also on parametric representations. Parametric representations are used also in spectral theory of linear operators (see [3], [11]).

Main theorem of this note lead to parametric representations of our new area Nevanlinna type spaces.

We introduce now another infinite product which is the main object of this note. It is known that (see [3]) the following assertion is true. The infinite Blaschke type product $B_{\alpha}(z,\{z_k\}), \alpha > -1$

$$B_{\alpha}(z,\{z_k\}) = \prod_{k=1}^{\infty} \left(1 - \frac{z}{z_k}\right) \exp\left(-W_{\alpha}(z,z_k)\right), \text{ and}$$

$$W_{\alpha}(z,\xi) = \sum_{k=1}^{\infty} \frac{\Gamma(\alpha+k+2)}{\Gamma(\alpha+2)\Gamma(k+1)} \times$$
$$\times \left(\left(\bar{\xi} z\right)^k \int_{|\xi|}^1 \frac{(1-x)^{\alpha} dx}{x^{k+1}} - \left(\frac{z}{\xi}\right)^k \int_0^{|\xi|} (1-x)^{\alpha} x^{k-1}\, dx \right), z, \xi \in \mathbf{D},$$

is converges uniformly within **D** if and only if $\sum_{k=1}^{\infty}(1-|z_k|)^{\alpha+1}<\infty$. Moreover it represents an analytic function in **D**.

2. MAIN RESULTS

We formulate now main results of this paper.

Theorem. *Let* $\alpha>-1$, $p\in(0,\infty)$ *and* $\{z_k\}_{k=1}^{\infty}$ *be a sequence of complex numbers in the unit disk* $\boldsymbol{D}$ *such that* $0<|z_k|\leq 1, k=1,\dots$

1) *Let* $\int_0^1(1-r)^{\alpha+p}\,n^p(r)dr<\infty$.

Then we have $B_t(z,\{z_k\})\in N_{\alpha,\beta}^{\infty,p}$, $\alpha>-1$, $\beta\geq 0$

if $0<p\leq 1$, $\frac{\alpha+1}{p}<t<2+\frac{\alpha+1}{p}$,

if $1<p<\infty$, $1+\frac{\alpha}{p}<t<2+\frac{\alpha+1}{p}$

and the infinite product converges absolutely and uniformly within D.

2) *Let* $\int_0^1(1-r)^{\alpha+1}\,n(r)dr<\infty$.

Then we have $B_t(z,\{z_k\})\in N_{\alpha,\beta}^{p},\alpha>-1,\beta>-1,t\in(\alpha+1,\alpha+3)$ *and the infinite product converges absolutely and uniformly within D.*

3) *Let* $\int_0^1(1-r)^{\gamma}\,n(r)dr<\infty$.

Then we have $B_t(z,\{z_k\})\in N_{p,\gamma,\nu}$, $\nu>-1$, $\gamma>0$, $t\in(\gamma,\gamma+2)$ *and the infinite product converges absolutely and uniformly within D.*

Proofs of our assertions are based on the following propositions.

Proposition 1. *Let* $f\in H(\boldsymbol{D})$, $\beta>-1,\gamma\geq 0$, $0<q<\infty$. *Then*

$$\left(\int_0^1(1-\tau)^{\beta+(\gamma+1)q}\left(\int_{\boldsymbol{T}}\log^+|f(\tau\xi)|dm(\xi)\right)^q d\tau\right)^{\frac{1}{q}}\leq$$

$$\leq c_1\left(\int_0^1(1-\tau)^{\beta}\left(\int_{|z|<\tau}\log^+|f(z)|(1-|z|)^{\gamma}dm_2(z)\right)^q d\tau\right)^{\frac{1}{q}}.$$

Proposition 2. *Let* $f\in H(D), p\geq 1$, $q\in(0,\infty)$, $\alpha>-1$, $\beta>-1,\tau=\beta+\frac{q}{p}(\alpha+1)$. *Then*

$$\int_0^1\left(\int_{|z|<R}(\log^+|f(z)|)^p\,(1-|z|)^{\alpha}dm_2(z)\right)^{q/p}(1-R)^{\beta}\,dR<\infty$$

if and only if

$$\int_0^1\left(\int_{\boldsymbol{T}}(\log^+|f(r\xi)|)^p\,dm(\xi)\right)^{q/p}(1-|z|)^{\tau}\,dr<\infty.$$

We define an extention of $T(f,r)$ to $T_q(f,r)$ in a usual way (see [10-11]).

Proposition 3. *Let* $f\in H(D)$, $q\geq 1$ *and* $p\leq s$. *Then*

$$\left(\int_0^1 T_q^s\,(r,f)(1-|z|)^{\alpha} d|z|\right)^{p/s} \leq$$

$$\leq\; c_5 \int_0^1 (1-r)^{\tau} \left(\sup_{0<p<r} T_q\,(\rho,f)(1-|\rho|)^{\gamma}\right)^{p} dr,$$

for the following values of indexes:

$$\alpha > -1,\;\; p,\;\; q,\; s \in (0,\infty), \gamma \geq 0, \tau = (\alpha+1)(p/s) - \gamma p - 1.$$

Our assertions in particular extend certain results obtained earlier in [12] and they lead directly to parametric representations of large area Nevanlinna spaces which we introduced in this note. We will discuss this in our next papers.

Related interesting problems in the half plane may also be considered (see for example [13] and various references there). Another complete version of this note with other results can be seen in [14] and with complete proofs.

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УДК 517.55+517.33 Научная статья

НЕКОТОРЫЕ ЗАМЕЧАНИЯ К ПРОИЗВЕДЕНИЯМ ТИПА БЛЯШКЕ В ПРОСТРАНСТВАХ ТИПА НЕВАНЛИННА В ЕДИНИЧНОМ КРУГЕ*

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Аннотация. В работе вводятся новые пространства типа Неванлинна в единичном круге и приводятся новые утверждения о принадлежности бесконечных произведений типа Бляшке этим пространствам. Ранее подобные теоремы для этих же бесконечных произведений были доказаны различными авторами в менее общих классах типа Неванлина в единичном круге. Наши теоремы могут быть в дальнейшем применены для получения новых параметрических представлений указанных новых широких классов типа Неванлинна в единичном круге через упомянутые нами бесконечные произведения типа Бляшке.

Ключевые слова: аналитическая функция, произведение типа Бляшке, пространства типа Неванлинна, характеристика типа Неванлинна, параметрические представления

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